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Differential Structure of Space-Time and Its Prolongations to Singular Boundaries

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After briefly presenting basic ideas of the differential space theory, the global description of space-times in terms of this theory is given. The space-times of a straight cosmic string and that of the closed Friedman world model are our standard examples. A space-time with its singular boundary is no longer a differentiable manifold, but it can be organized into a differential space, and the differential structure of space-time can be prolonged to its singular boundary. In general, the procedure is not unique. A simple classification of differential structure prolongations is also presented.

INTRODUCTION

The growing interest in quantum gravity has encouraged both the search for generalizations of the manifold concept and improvements of global methods related to traditional space-times. Mathematical theories which are simultaneously global and more general than the manifold theory have been known to mathematicians for some time (e.g., Aronszajn, 1967; Spallek, 1969; Sikorski 1967, 1971; MacLane, 1970; Marshall, 1975; Mostov, 1979), but so far they have not found more extensive applications in physics. In the present paper, we continue our previous research (Gruszczak *et al.*, 1988, 1989; Heller *et al.*, 1989) to model the physical space-time with the help of Sikorski's differential space theory. The advantages of this theory are its relative simplicity and close analogy to the standard theory of differentiable manifolds.

In the present work we focus on two major issues. The first issue is a global description of space-times met in the orthodox theory of general relativity. The space-time of the straight cosmic string and that of the closed

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Friedman cosmological model will serve as our standard examples. However, our methods apply to other space-times as well.

The second issue dealt with in the present work is an attempt to apply the same methods of investigation to analyze relativistic space-times with their singular boundaries (singularities are treated in the classical way, i.e., quantum gravity effects are not taken into account). One should notice that space-time with its singular boundary is no longer a differentiable manifold, but it can be organized into a differential space (usually in many nonequivalent ways). It turns out that singular boundaries can have unexpected differential properties. For instance, topologically quite innocent singularities (such as "regular singularities" which essentially originate by cutting off some parts of space-time) admit such extensions of the differential structure from the space-time into themselves that the differential space in question (modeling space-time with its singularities) cannot be embedded in $\mathbb{R}^n$ for any $n<\infty$. We effectively construct the background differential spaces of the cosmic string and the closed Friedman model with their respective singularities—the so-called quasiregular singularity in the case of the cosmic string, and the curvature singularities in the case of the closed Friedman model. This part of our research is a continuation of Gruszczak (1990), Gruszczak *et al.*, (1991), and Heller (1992). We also give a simple classification of possible prolongations of differential structures onto singular boundaries.

We organize our material in the following way. In Section 1 we give, for the reader's convenience, a short account of the theory of differential spaces. In Section 2 we effectively construct two space-time models (those of a cosmic string and the closed Friedman worlds) in terms of differential spaces. Section 3 contains a methodological discussion concerning the classical singularity problem, which allows us, in Section 4, to define prolongations of differential structures to singularities. In Section 5 we discuss the Cauchy singular boundary problem and in Section 6 introduce the Cauchy completion of metric differential spaces. Then, in Section 7, we come back to our two standard examples (cosmic string and closed Friedman models) and construct for them prolongations of differential structures to their singular boundaries. A simple classification of differential structure prolongations is presented in Section 8, and finally, in Section 9, our main results are summarized. Two appendices contain more technical information of the theory of differential spaces which is in our constructions: Appendix A deals with topology and Appendix B with the (local) differential dimension of differential spaces.

1. BASIC THEORY OF DIFFERENTIAL SPACES

In all major physical theories space-time is modeled by differentiable manifolds on which other structures characteristic for a given theory are superimposed. In this context we shall speak of the *background manifold* of a given space-time. The fact that such background manifolds are usually defined in terms of maps and atlases causes some unnecessary complications as far as global investigations of space-time are concerned. In what follows we propose to replace the differentiable manifold concept by the differential space concept (d-space for short) and, correspondingly, to speak of the *background differential space* of a given space-time. The theory of d-spaces is global from the very beginning (it does not require the notions of maps and atlases), and is by far more general than that of differentiable manifolds. One should keep in mind that differentiable manifolds are special cases of d-spaces and in standard physical theories, such as general relativity, essentially nothing changes if we adopt the conceptual machinery of d-space theory to work with space-times, but d-spaces which are not differentiable manifolds might be needed in some approaches to quantum gravity (where the manifold structure of space-time is expected to break down) or when discussing the classical singularity problem in relativistic cosmology and astrophysics.

The idea underlying the theory of d-spaces is the following. The family $\mathscr{C}(X)$ of real-valued functions on a "reasonable" (e.g., Hausdorff) topological space X forms a linear algebra, and the sets $\mathscr{C}_x \subset \mathscr{C}(X)$ of functions that vanish at x, where x is any point of X, form its maximal ideals. Since the space of such maximal ideals is isomorphic to X, one can reconstruct the topological and some geometric properties of X from the knowledge of algebraic properties of $\mathscr{C}(X)$. In particular, the family $C^\infty(M)$ of all smooth real-valued functions on a differentiable manifold M is a linear algebra, and the manifold structure of M can be reconstructed from the structure of $C^\infty(M)$ (Geroch, 1972). The d-space theory is one of a few possible ways to implement the above ideas into a computational scheme. In the rest of this section we shall present its main concepts.

Let M be any set, and $\mathscr{C}$ a family of real functions on M. The weakest topology on M in which all functions of $\mathscr{C}$ are continuous will be denoted by $\tau_{\mathscr{C}}$.

A function $f: A \to \mathbb{R}$, where $A \subseteq M$ is a subset of a topological space $(M, \tau_{\mathscr{C}})$, is said to be a *local $\mathscr{C}$-function* if for any point $p_0 \in A$ there exists a neighborhood B_0 of p_0 in the topological space $(A, \tau_{\mathscr{C}}|A)$ and a function $g \in \mathscr{C}$ such that $f|B_0 = g|B_0$. The set of all local $\mathscr{C}$-functions on $A \subseteq M$ will be denoted by $\mathscr{C}_A$. In general, $\mathscr{C}|A \subset \mathscr{C}_A$; if, however, $\mathscr{C} = \mathscr{C}_M$ (for $A = M$), $\mathscr{C}$ is said to be *closed with respect to localization.*

Let us define $\mathrm{sc}\mathscr{C} := \{\omega \circ (\alpha_1, \ldots, \alpha_n)\colon \alpha_1, \ldots, \alpha_n \in \mathscr{C},\ \omega \in C^\infty(\mathbb{R}^n),\ n \in \mathbb{N}\}$. The family $\mathscr{C}$ is said to be *closed with respect to superposition with smooth functions ω* on $\mathbb{R}^n$ if $\mathscr{C} = \mathrm{sc}\,\mathscr{C}$.

Definition 1.1. A family $\mathscr{C}$ of real functions defined on a set M is said to be a *differential structure on M* if:

(a) $\mathscr{C}$ is closed with respect to localization, $\mathscr{C}=\mathscr{C}_M$.

(b) $\mathscr{C}$ is closed with respect to superposition with smooth functions on $\mathbb{R}^n$, i.e., if $\mathscr{C}=(\mathrm{sc}\,\mathscr{C})_M$.

The pair $(M,\mathscr{C})$, where $\mathscr{C}$ is a differential structure on M, is called a *differential space* (*d-space*).

If (M, $\mathscr{A}$) is a C^∞-manifold, where $\mathscr{A}$ is an atlas on M, the family $C^\infty(M)$ of all C^∞-functions on M is a differential structure on M. The pair $(M, C^\infty(M))$ is a *d*-space which is locally diffeomorphic (see below) to $\mathbb{R}^n$.

Let $(M,\mathscr{C})$ be a d-space and A any subset of M. $(A,\mathscr{C}_A)$ is a d-space; it will be called a *differential subspace* of $(M,\mathscr{C})$ [*d-subspace* of $(M,\mathscr{C})$, for short]. For example, if $M\subseteq\mathbb{R}^n$ and $\pi_1,\ldots,\pi_n$ are projections onto corresponding coordinates, then $(M,\mathscr{C})$, where

$$\mathscr{C}=(\mathrm{sc}(\{\pi_1|M,\ldots,\pi_n|M\}))_M$$

is a d-subspace of the d-space $(\mathbb{R}^n, C^\infty(\mathbb{R}^n))$. It can be shown that any subset of $\mathbb{R}^n$ can be organized into a d-space, but there are many d-spaces which are not subsets of $\mathbb{R}^n$ for any n.

Any family of real functions on a set M can be supplemented to form a differential structure. Let $\mathscr{C}_0$ be a family of real functions on M. The weakest (in the sense of inclusion) differential structure $\mathscr{C}$ on M containing $\mathscr{C}_0$ is said to be *generated* by $\mathscr{C}_0$; we write $\mathscr{C}=\mathrm{Gen}\,\mathscr{C}_0:=(\mathrm{sc}\,\mathscr{C}_0)_M$. For example, the Euclidean differential structure $\mathscr{E}_n$ on the manifold $\mathbb{R}^n$ is generated by the family of projections onto the axes: $\mathscr{E}_n:=C^\infty(\mathbb{R}^n)=\mathrm{Gen}\{\pi_i\colon\mathbb{R}^n\to\mathbb{R},\ i=1,\ldots,n\}$. If a d-space is generated by a finite number of functions it is said to be *finitely generated* (Sasin and Żekanowski, 1987).

$\mathbb{R}^n$ spaces are "modeling spaces" for finitely generated d-spaces. The following result will be useful in our considerations: Let $(M,\mathscr{C})$ be a Hausdorff d-space with $\mathscr{C}=\mathrm{Gen}\{f_1,\ldots,f_n\}$. The mapping $F\colon M\to\mathbb{R}^n$, $F:=(f_1,\ldots,f_n)$, is a diffeomorphism onto the image $(F(M),(\mathscr{E}_n)_{F(M)})$ (Sasin and Żekanowski, 1987). In the theory of d-spaces the diffeomorphism notion is defined in the following way. Let $(M,\mathscr{C})$ and $(N,\mathscr{D})$ be d-spaces. A mapping $f\colon M\to N$ is said to be *smooth* if $\alpha\circ f\in\mathscr{C}$ for every $\alpha\in\mathscr{D}$. Every mapping $f\colon M\to N$ from a d-space $(M,\mathscr{C})$ to a d-space $(N,\mathscr{D})$ is a *diffeomorphism* if it is a bijection and both f and f^{-1} are smooth mappings.

The totality of d-spaces, as objects, with smooth mappings between them (with ordinary composition of mappings), as morphisms, constitute the *category of d-spaces*. The category of differential manifolds is a subcategory of the category of d-spaces.

For a differential space $(M,\mathscr{C})$, we define a *tangent vector* to M at $p\in M$ as a linear mapping $v\colon\mathscr{C}\to\mathbb{R}$ such that

$$v(fg)=v(f)g(p)+f(p)v(g),\qquad p\in M,\quad f,g\in\mathscr{C}$$

The set of all tangent vectors to M at p is called *tangent space* (denoted by T_pM) to the d-space $(M, \mathscr{C})$ at p. The space T_pM is a linear subspace of the linear space $\mathbb{R}^{\mathscr{C}}$ of all real functions defined on $\mathscr{C}$ (see, for example, Isham, 1989). A *vector field* on a d-space $(M, \mathscr{C})$ is a mapping

$$V: \quad M \ni p \to V(p) \in T_pM \subset \bigcup_{q \in M} T_qM$$

A vector field V on M is said to be *smooth* if for every $f \in \mathscr{C}$ the real function $V(\cdot)(f): M \ni p \to V(p)(f) \in \mathbb{R}$ is an element of $\mathscr{C}$.

Some d-spaces can be regarded as "parts" of a differentiable manifold; they are called d-spaces of class D_0 and are simple to deal with. They will play an important role in our further analysis. A d-space $(M, \mathscr{C})$ is said to be of *class* D_0 if for every point $p \in M$ there exist an open neighborhood U *of* p and a differentiable manifold N such that $U \subset N$, $\dim N = \dim T_pM$, and $C^\infty(U) = C^\infty(N)_U$.

Let us note that the dimensionality of d-spaces is not a part of their definition, but it can be additionally defined. The theory of differential dimension is well elaborated (Multarzyński and Sasin, 1989; Heller and Sasin, 1990). In the present work we shall distinguish two differential dimension concepts: (1) the *local differential dimension* at a point $p \in M$ is simply the dimension of the tangent space T_pM to $(M, \mathscr{C})$; (2) the (*global*) *differential dimension* is the real number n such that (a) $n = \dim T_pM$ for every $p \in M$, and (b) for every $p \in M$ and every vector $v \in T_pM$ there is a smooth tangent vector field V on $(M, \mathscr{C})$ such that $V(p) = v$. Those facts of the theory of differential dimension which will be used in the present work are summarized in Appendix B.

For a full theory of differential spaces the reader should consult the original monograph by Sikorski (1972) or the review paper by Heller *et al.* (1989).

2. d-SPACE DESCRIPTION OF SPACE-TIME

In our previous works (quoted in the Introduction) we have developed theoretical aspects of modeling space-times on d-spaces rather than on smooth manifolds. In the present section we present an "algorithm" which will allow one to effectively apply d-space methods to analyze space-times met in general relativity. We do that by presenting two, typical in certain aspects, examples.

Example 2.1. Differential Space of a Cosmic String

Recent fascination with cosmic strings as products of phase transitions in the very early universe (e.g., Kibble, 1976; Vilenkin, 1981*a,b*; Hiscock, 1985; Gott, 1985) revived interest in space-time with the metric

$$ds^2 = -dt^2 + dr^2 + r^2\, d\theta^2 + dz^2 \tag{2.1}$$

where t, $z \in (-\infty, \infty)$, $r \in (0, \infty)$, $\theta \in \langle 0, 2\pi - \Delta)$, $\Delta \in \langle 0, 2\pi)$. This space-time appeared in Staruszkiewicz (1963), where it was interpreted—within the context of the three-dimensional theory of gravity—as the gravitational field of a point mass, and can be traced back to Levi-Civita (1917). Today it is regarded as the limit of space-times the source of which consists of a filament situated parallel to the z axis, or as a space-time with the corresponding distributional type of energy-momentum tensor. The singularity appearing in this space-time is a typical example of quasiregular singularities. The class of such singularities was extensively analyzed by Ellis and Schmidt (1977). Quasiregular singularities are of a topological nature and, although they cannot be removed by simply extending the space-time, the components of the curvature tensor behave perfectly well when propagating along curves terminating at the singularity. In our case, the space-time is flat (Minkowskian) and it looks as if the whole curvature were concentrated in the singularity (which is also called the *conic singularity*).

To find the background d-space of this space-time let us consider a four-dimensional hypersurface immersed in $\mathbb{R}^5$,

$$C^{(4)} = \{p \in \mathbb{R}^5 : [(z^1)^2 + (z^2)^2]^{1/2} = a^{-1} z^4\}$$

where $p = (z^0, z^1, z^2, z^3, z^4)$, $a \in \mathbb{R}$. The set of singular points of this hypersurface has the form

$$S = \{p \in \mathbb{R}^5 : p = (z^0, 0, 0, z^3, 0), z^0, z^3 \in \mathbb{R}\}$$

It can be easily seen that metric (2.1) on $C^{(4)} - S$ is induced by the metric $\eta^{(5)} = \mathrm{diag}(-1, 1, 1, 1, 1)$ on $\mathbb{R}^5$. Since the conic singularity is given explicitly, there is no need of any special definition.

The manifold $C^{(4)} - S$ can be described as a finitely generated d-space. Let $\tilde{P} := \mathbb{R}^2 \times (0, \infty) \times \langle 0, 2\pi \rangle$ be a "parameter space," and let us parametrize the hypersurface $C^{(4)}$ with the help of the functions $\tilde{\alpha}_i : \tilde{P} \to \mathbb{R}$, $i = 0, 1, \ldots, 4$:

$$\begin{aligned} \tilde{\alpha}_0(q) &:= z^0 = t \\ \tilde{\alpha}_1(q) &:= z^1 = \rho \cos \phi \\ \tilde{\alpha}_2(q) &:= z^2 = \rho \sin \phi \\ \tilde{\alpha}_3(q) &:= z^3 = z \\ \tilde{\alpha}_4(q) &:= z^4 = a\rho \end{aligned} \tag{2.2}$$

for $q=(t, z, \rho, \phi)\in\tilde{P}$, where

$$r=\rho(a^2+1)^{1/2}, \qquad \theta=\phi(a^2+1)^{-1/2}$$

Let $\tilde{\mathscr{P}}$ be the differential structure on $\tilde{P}$ generated by $\{\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_4\}$, i.e., $\tilde{\mathscr{P}}=\mathrm{Gen}\{\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_4\}$. It turn out that the d-space $(\tilde{P}, \tilde{\mathscr{P}})$ is not Hausdorff; indeed, the functions $\tilde{\alpha}_i$, $i=0, 1, \ldots, 4$, do not distinguish the points $(t, z, \rho, 0)$ and $(t, z, \rho, 2\pi)$. To cure this situation, let us define the so-called *Hausdorff equivalence relation* ρ_{H} in the following way: for any $q_1, q_2\in\tilde{P}$, $q_1\rho_{\mathrm{H}}q_2$ if and only if $\tilde{\alpha}_i(q_1)=\tilde{\alpha}_i(q_2)$, $i=0, 1, \ldots, 4$. Now, let $\mathscr{P}:=\mathrm{Gen}\{\alpha_0, \alpha_1, \ldots, \alpha_4\}$ be the differential structure on $P:=\tilde{P}/\rho_{\mathrm{H}}$ (Sasin, 1988). $\mathscr{P}$ is finitely generated by the family of functions $\{\alpha_0, \alpha_1, \ldots, \alpha_4\}$ given by $\alpha_i([p]):=\tilde{\alpha}_i(p)$, for $p\in\tilde{P}$, $[p]\in P$, $i=0, 1, \ldots, 4$, and is sometimes denoted by $\tilde{\mathscr{P}}/\rho_{\mathrm{H}}$; one has $\mathscr{P}=\tilde{\mathscr{P}}/\rho_{\mathrm{H}}$.

Now, we can formulate our final result in the following form.

Proposition 2.1. The d-space $(P, \mathscr{P})$ is diffeomorphic to the d-space $(C^{(4)}-S, (\mathscr{E}_5)_{C^{(4)}-S})$, which is a differential subspace of $(\mathbb{R}^5, \mathscr{E}_5)$, i.e., $(P, \mathscr{P})$ is diffeomorphic to the background manifold of a cosmic string.

Proof. The mapping $F\colon P\to\mathbb{R}^5$, $F:=(\alpha_0, \alpha_1, \ldots, \alpha_4)$, is a diffeomorphism of $(P, \mathscr{P})$ onto the image $(F(P), (\mathscr{E}_5)_{F(P)})$. By direct computation one can see that $F(P)=C^{(4)}-S$. It follows that

$$F(P, \mathscr{P})\to(C^{(4)}-S, (\mathscr{E}_5)_{C^{(4)}-S})$$

is the diffeomorphism onto the d-subspace of $(\mathbb{R}^5, \mathscr{E}_5)$ (see Section 1). ■

Space-times with quasiregular singularities were investigated by Vickers (1985, 1987, 1990) with the help of traditional, although sophisticated and rather tedious, methods. Among other results, he was able to show that only directions normal to the quasiregular singularity are degenerate, whereas those tangent to the singularity are well behaved (Vickers, 1990). It is interesting to note that such nondegenerate directions (or vector fields) which go smoothly through the singularity can be easily constructed by using the d-space formalism. For instance, the vector fields V, $W\colon\mathscr{P}\to\mathscr{P}$ given by the conditions $W(\alpha_i)=\delta_{i0}$ and $V(\alpha_i)=\delta_{i3}$, $i=0, 1, \ldots, 4$, are smooth in spite of the fact that at the singularity the dimension jumps from 4 to 5 (Gruszczak *et al.*, 1991).

Example 2.2. Differential Space of the Closed Friedman Universe

As our second example we shall consider the closed Friedman universe filled with radiation. Its metric has the form

$$ds^2=a^2(\eta)(-d\eta^2+d\chi^2+\sin^2\chi\,(d\theta^2+\sin^2\theta\,d\phi^2)) \tag{2.3}$$

where $a(\eta)=a_1 \sin \eta$, $\eta, \chi, \theta \in (0, \pi)$, $\phi \in \langle 0, 2\pi \rangle$, and a_1 is a constant. Let

$$\tilde{P} := \{p : p=(\eta, \chi, \theta, \phi) \in (0, \pi) \times \langle 0, \pi \rangle \times \langle 0, \pi \rangle \times \langle 0, 2\pi \rangle\}$$

be a "parameter space." An isometric embedding of the model in $(\mathbb{R}^5, \eta^{(5)})$ is given by

$$z_i : \quad \tilde{P} \to \mathbb{R}, \quad i=1, 2, \ldots, 5, \qquad (z_0, z_1, \ldots, z_4) \in \mathbb{R}^{(5)}, \qquad p \in \tilde{P}$$

where

$$\begin{aligned} z_0 &= a_1 \eta =: \tilde{\alpha}_0(p) \\ z_1 &= a_1 \sin \eta \cos \chi =: \tilde{\alpha}_1(p) \\ z_2 &= a_1 \sin \eta \sin \chi \cos \theta =: \tilde{\alpha}_2(p) \\ z_3 &= a_1 \sin \eta \sin \chi \sin \theta \cos \phi =: \tilde{\alpha}_3(p) \\ z_4 &= a_1 \sin \eta \sin \chi \sin \theta \sin \phi =: \tilde{\alpha}_4(p) \end{aligned} \tag{2.4}$$

The d-space $(\tilde{P}, \tilde{\mathscr{P}})$, $\tilde{\mathscr{P}} = \mathrm{Gen}\{\tilde{\alpha}_0, \tilde{\alpha}^1, \ldots, \tilde{\alpha}_4\}$, is not Hausdorff; indeed, $\tilde{\alpha}_i(\eta, \chi, \theta, 0) = \tilde{\alpha}_i(\eta, \chi, \theta, 2\pi)$, $i=0, 1, \ldots, 4$.

As in the previous example, we introduce $P := \tilde{P}/\rho_{\mathrm{H}}$ and $\mathscr{P} = \mathrm{Gen}\{\alpha_0, \alpha_1, \ldots, \alpha_4\}$, $\alpha_i([p]) := \tilde{\alpha}_i(p)$, for $p \in \tilde{P}$, $[p] \in P$. Again we can formulate the following result.

Proposition 2.2. $(P, \mathscr{P})$ is diffeomorphic to the background manifold of the closed Friedman world model with radiation.

Proof. The proof is similar to that of Proposition 2.1. ■

As shown by Clarke (1970), every Lorentz manifold can be isometrically embedded in $\mathbb{R}^n$ with n sufficiently large. Therefore, the above procedure of constructing d-spaces diffeomorphic to a given space-time background manifold can be repeated, in principle, for every space-time. As we have seen, the method is global from the very beginning. Moreover, the case of the cosmic string suggests that at least some singularities could be analyzed by using the theory of d-spaces. This poses the question: can differential structure be prolonged from the background manifold of a given space-time to its singular boundary? We shall deal with this problem in the following sections.

3. METHODOLOGY OF THE CLASSICAL SINGULARITY PROBLEM

The physical observer perceives reality as a system of objects entangled in a net of manifold relations some of which form the space-time arena. It seems, therefore, that a good definition of a singular boundary of space-time—if it is to be given from the observer's perspective—should be formulated in terms of relations and objects determined from the inside of space-time. Taking into account this strategy, the following methodology has been elaborated when dealing with the problem of singularities in general relativity (Hawking and Ellis, 1973; Tipler *et al.*, 1980):

1. One defines a singular boundary $\tilde{\partial}M$ of space-time M as a set, for the time being devoid of any structure, of (usually timelike) incomplete curves (geodesic-incomplete, bounded acceleration incomplete, or b-incomplete); the incompleteness of curves is treated as a symptom of the existence of singularities.
2. An equivalence relation $\rho \subset \tilde{\partial}M \times \tilde{\partial}M$ is defined which divides $\tilde{\partial}M$ into the classes of curves $[\gamma]$, $\gamma \in \tilde{\partial}M$, each class defining the same "ideal" point, i.e., a point of the singular boundary ∂M of space-time M; $\partial M = \tilde{\partial}M/\rho$.

Various definitions of ∂M (treated as a set) differ in the choice of $\tilde{\partial}M$ or in the choice of ρ. In order to make the construction interesting from the physical point of view, one must introduce an additional structure on $M \cup \partial M$, namely a topological structure. Without a suitable toplogy, ∂M has no "contact" with the space-time M. There is, however, a general requirement concerning topology on $M \cup \partial M$. It must be such that

$$M \text{ is dense in } M \cup \partial M \tag{3.1}$$

This assumption ensures that every point of ∂M is attainable from the inside of space-time.

During the last 15 years, much work has focused on the problem of a suitable topology on $M \cup \partial M$. The problem has turned out to be difficult; even the physically promising and mathematically elegant construction proposed by Schmidt (1971) led to physically nonacceptable non-Hausdorff topologies (Bosshard, 1976; Johnson, 1977). In such a situation even partial solutions are valuable. Such partial solutions (valid for restricted classes of space-times) are, for instance, those proposed by Clarke (1978), Dodson (1978, 1979), and the present authors (Gruszczak, 1990; Gruszczak *et al.* 1991).

In the present paper we propose an enrichment, and the next logical step, of the above-discussed strategy. A structure richer than topology is differential structure; we shall discuss how to prolong the differential structure from the background d-space of a given space-time to its singular boundary so that the space-time *with singularities* can be organized into a d-space, and the methods of the d-space theory can be used to investigate the singularity problem. Since any differential structure $\mathscr{C}$ determines the topology $\tau_{\mathscr{C}}$, the prolongation must be done in such a way that the correct topology on $M \cup \partial M$ is preserved.

The question arises whether every space time with its singular boundary can be organized into a d-space. It turns out that the answer to this question is either positive or negative depending on the way one defines the singular boundary in question.

Example 3.1. Closed Friedman World Model with b-Boundary

It has been shown by Johnson (1977, Theorem 2) that if ∂_b is the part of the b-boundary of M which is defined by b-incomplete curves along which the usual radial coordinate r is not bounded away from zero (in such a case, ∂_b is called the *essential boundary of* M), then ∂_b consists of one point x_0 and the only neighborhood of x_0 in M is M itself. On the other hand, it is known from the theory of d-spaces (see Appendix A) that, for any d-space $(M, \mathscr{C})$, the topological space $(M, \tau_{\mathscr{C}})$ is always $\mathscr{C}$-regular, and if it is T_0, then it must be $T_{1/3}$. Let $(\bar{M}, \tau)$ be a topological space such that $\bar{M} = M \cup \{x_0\}$; $x_0 \notin M$ and M is a differentiable manifold. Let us assume that $\bar{\mathscr{C}}$ is a differential structure on $\bar{M}$ such that $\tau_{\bar{\mathscr{C}}} = \tau$. Evidently, $\tau_{\bar{\mathscr{C}}}|M$ is a T_0-topology. However, the only neighborhood of x_0 in M is M itself, which implies that for every $f \in \bar{\mathscr{C}}$ and for every $x \in M, f(x) = f(x_0)$. Therefore, $\bar{\mathscr{C}} = \{k : k \in \mathbb{R}\}$ and consequently $\tau_{\bar{\mathscr{C}}}|M$ is not T_0. It follows that there is no differential structure $\bar{\mathscr{C}}$ on $\bar{M}$ such that $\tau = \tau_{\bar{\mathscr{C}}}$.

From the above it follows that there is no differential structure $\mathscr{C}$ on the b-completion of the closed Friedman space-time M; the reason for this is the inconsistency of topologies. In other words, the closed Friedman space-time with its b-boundary is not a d-space, and it cannot be embedded in any smooth manifold of no matter how many dimensions (Heller *et al.*, 1992).

If one tries to cure the situation by introducing a suitable equivalence relation [which is a standard method in such cases (Sasin, 1988)], it turns out that all points of M must be glued together to obtain a Hausdorff space.

The same situation occurs in the Schwarzschild solution with its singularity regarded as a b-boundary (Johnson, 1977).

We can see that the reason for the nonexistence of a differential structure on a space-time with its singular boundary is the "noncompatibility of topologies," and this should be regarded as a symptom of the fact that the corresponding singular boundary has been defined in an unsuitable manner (there is a general consensus that this is indeed the case as far as the b-boundary is concerned). If there is no "noncompatibility of topologies," the differential structure can be prolonged from the background d-space of space-time onto its singular boundary. We discuss this in the next section.

4. PROLONGATION OF DIFFERENTIAL STRUCTURES TO SINGULARITIES

Let $(M, \mathscr{C})$ be the background d-space of a space-time M (i.e., a manifold) and ∂N its singular boundary [in the following, unless directly specified, we shall have in mind any singular boundary satisfying condition (3.1)].

Definition 4.1. A d-space $(\bar{N}, \bar{\mathscr{C}})$ is said to be a *prolonged d-space* of a given background d-space of space-time $(M, \mathscr{C})$ to its singular boundary ∂N (*prolonged d-space* for short) if there exists a d-space $(N, \tilde{\mathscr{C}})$ diffeomorphic to $(M, \mathscr{C})$ such that:

(a) $\bar{N} = \partial N \cup N$, where N is dense in $\bar{N}$.
(b) $(N, \bar{\mathscr{C}}_N) = (N, \tilde{\mathscr{C}})$.

$\bar{\mathscr{C}}$ is said to be a *prolonged differential structure* of the differential structure $\mathscr{C}$. We shall also speak of a *prolongation* of the differential structure (to the singular boundary).

Such a prolongation, if exists (see the previous section) need not be unique.

Example 4.1

Let $(\mathbb{R}, \mathscr{E}_1)$, $\mathscr{E}_1 = C^\infty(\mathbb{R})$, be a (one-dimensional) Euclidean d-space. Let further $(\mathbb{R}_0, \mathscr{C}_0)$ be a d-space such that $\mathbb{R}_0 = \mathbb{R} - \{0\}$ and $\mathscr{C}_0 = \operatorname{Gen}\{\alpha_0\}$, where $\alpha_0 = \mathrm{id}_{\mathbb{R}_0}$. Of course, $(\mathbb{R}, \mathscr{E}_1)$ is a prolonged d-space of $(\mathbb{R}_0, \mathscr{C}_0)$ to the singular boundary $\{0\}$ [$\{0\}$ is the so-called regular singularity in the terminology of Ellis and Schmidt (1977)]. This prolongation is natural in the sense that it restores the original Euclidean differential structure $\mathscr{E}_1$ lost by removing a point from $\mathbb{R}$. Indeed, one has $\mathbb{R} = \mathbb{R}_0 \cup \{0\}$ and $\mathscr{E}_1 = \operatorname{Gen}\{\alpha\}$. where $\alpha = \mathrm{id}_{\mathbb{R}}$.

However, the differential structure on $\mathbb{R}_0$ can be prolonged in a different way.

Example 4.2

Let $\mathscr{C}_2 = \{f: f: \mathbb{R} \to \mathbb{R}$ are functions such that $f \mid \mathbb{R}_0 \in \mathscr{C}_0\}$. $(\mathbb{R}, \mathscr{C}_2)$ is an infinitely generated d-space, $\mathscr{C}_2 \neq \mathscr{E}_1$, since to $\mathscr{C}_2$ belong (besides smooth functions from $\mathscr{E}_1$) nonsmooth (in the usual sense) functions, e.g., the function $|x|$. $(\mathbb{R}, \mathscr{C}_2)$ is also a prolongation of $(\mathbb{R}_0, \mathscr{C}_0)$. Let us note that the d-space $(\mathbb{R}, \mathscr{C}_2)$, being infinitely generated, cannot be embedded in any $\mathbb{R}^n$ for any finite n (Sasin and Żekanowski, 1987).

5. CAUCHY SINGULAR BOUNDARIES OF SPACE-TIME

Every Lorentz manifold (M, g) is a metrizable but not metric space. The point is that no Lorentz manifold has a uniquely defined uniform structure. This fact implies the existence of many nonequivalent Cauchy completions, and lies at the basis of various troubles connected with attaching a "reasonable" singular boundary to space-time. One way out of the problem would be to distinguish, possibly on physical grounds, a single uniform structure with the corresponding Cauchy completion as a candidate to represent space-time together with its "reasonable" singular boundary, which, in this case, could rightly be called a *Cauchy singular boundary* of space-time. The problem was extensively discussed by Gruszczak (1990).

In previous work (Gruszczak *et al.*, 1991) we proposed a criterion naturally distinguishing a single uniform structure for a restricted class of space-times, and thereby correctly defining their Cauchy singular boundaries. It turns out that, for a time-orientable space-time (M, g) with the Levi-Civita connection reducible to an $O(3)$ structure, one can naturally select a nonvanishing timelike vector field ξ on M and a Riemann (positively defined) metric g^+ on M. We showed that in such a case the Cauchy boundary of the metric space (M, g^+) consists of "endpoints" of b-incomplete curves in (M, g), and consequently it should be regarded as the *Cauchy singular boundary* of space-time (M, g). For space-times belonging to this class, the problem of differential structure prolongation from the interior of space-time to its Cauchy singular boundary has been solved. We shall also discuss, as one of our main examples, the closed Friedman world model, the connection of which is not reducible to $O(3)$ structure. Therefore, its Cauchy boundary [in a detailed way described by Gruszczak (1990)] has no obvious physical interpretation except for the fact that it remains in agreement with our expectations. For this reason, we shall treat the closed Friedman model as a test of our method rather than a physically well-founded example. It will fulfill for us an important function of showing that if we define the singular boundary as a Cauchy boundary, independently of whether the singularity is a curvature singularity or not, our procedure of differential structure prolongation works well.

In the next section we present the construction of the Cauchy completion of metric d-spaces.

6. CAUCHY COMPLETION OF METRIC d-SPACES

In the present section we define metric d-spaces in general and construct their Cauchy boundaries in such a way that the differential structure on the Cauchy completion of the metric d-space will turn out to be the prolongation of the corresponding differential structure of the considered metric d-space.

Definition 6.1. The triple $(M, \mathscr{C}, d)$ will be called a *metric d-space* if:

(a) $d: M \times M \to \mathbb{R}_0^+$ is a distance function (in the usual sense)
(b) $\tau_{\mathscr{C}} = \tau_d$, i.e., the topology $\tau_{\mathscr{C}}$ is equivalent to the metric topology τ_d.
(c) $(M, \mathscr{C})$ is a d-space.

Any $(M, \mathscr{C}, d)$ will be called an *incomplete metric d-space* if (M, d) is incomplete as a metric space. Otherwise, $(M, \mathscr{C}, d)$ will be called a *complete metric d-space*. It is well known that every incomplete metric space (M, d) can be completed. The question is whether this remains true for incomplete metric d-spaces.

In order to establish the notation, let us briefly review the well-known completion procedure for metric spaces. In what follows the letter c always denotes a Cauchy sequence, $c = \{x_n, n \in \mathbb{N}\}$.

Let (M, d) be a metric space, and C and CM sets of Cauchy sequences of points from M defined in the following way:

$$C := \{c \subset M: c = \{x_n\}, x_n \in M, \forall n \in \mathbb{N}\}$$

$$CM \subset C, \qquad CM := \{c \subset M: \lim x_n \in M\}$$

Let $\hat{d}$ denote the distance function introduced in C or CM in the usual manner, i.e.,

$$\hat{d}(c^1, c^2) := \lim d(x_n^1, x_n^2), \qquad c^1, c^2 \in C \text{ or } CM$$

Two Cauchy sequences $c^1, c^2 \in C$ are equivalent ($c^1 \rho c^2$) if and only if $\hat{d}(c^1, c^2) = 0$.

Let us consider two metric spaces $(\bar{C}, \bar{d})$ and $(\overline{CM}, \bar{d})$, where $\bar{C} := C/\rho$, $\overline{CM} := CM/\rho$, and $\bar{d}([c^1], [c^2]) = \hat{d}(c^1, c^2)$, and the symbol $[\cdot]$ denotes the equivalence class with respect to the relation ρ.

One has (by construction):

(a) $(\bar{C}, \bar{d})$ is (by definition) a Cauchy completion of (M, d).
(b) $(\overline{CM}, \bar{d})$ is isometric to (M, d).
(c) $(\overline{CM}, \bar{d})$ is dense in $(\bar{C}, \bar{d})$.

The set $\partial M := \bar{C} - \overline{CM}$ will be called the *Cauchy boundary* of the metric d-space.

Let $(M, \mathscr{C}, d)$ be a metric d-space, $\overline{CM}$ the set of Cauchy sequences as above, and $i: M \to \overline{CM}$, given by $i(p) = [c]_p$, the isometry mentioned in (b), where $[c]_p$ is the class of all Cauchy sequences $\{x_n\} \subset M$ such that $x_n \to p \in M$.

Moreover, let us introduce the set $\tilde{\mathscr{C}}$ defined in the following way:

$$\tilde{\mathscr{C}} = \{\tilde{\alpha} : \overline{CM} \to \mathbb{R} : \tilde{\alpha} = \alpha \circ i^{-1}, \alpha \in \mathscr{C}\}$$

Proposition 6.1. The triple $(\overline{CM}, \tilde{\mathscr{C}}, \bar{d})$ is a metric d-space.

Proof. (1) Since i is a homeomorphism of the topological spaces $(M, \tau_{\mathscr{C}})$ and $(\overline{CM}, \tau_{\tilde{\mathscr{C}}})$ and it is an isometry of the metric spaces (M, d) and $(\overline{CM}, \bar{d})$, therefore the topologies $\tau_{\bar{d}}$ and $\tau_{\tilde{\mathscr{C}}}$ on $\overline{CM}$ are equivalent.

(2) It is easy to show that $\tilde{\mathscr{C}}$ is a d-structure on $\overline{CM}$. ■

Proposition 6.2. $(\overline{CM}, \tilde{\mathscr{C}})$ is diffeomorphic to $(M, \mathscr{C})$.

Proof. It is easy to check that $i: M \to \overline{CM}$ is a diffeomorphism of $(\overline{CM}, \tilde{\mathscr{C}})$ onto $(M, \mathscr{C})$ (see definition in Section 1). ■

Now let us come back to our discussion of metric d-spaces.

Definition 6.2. Every metric d-space $(\bar{C}, \bar{\mathscr{C}}, \bar{d})$ which satisfies the conditions (a) $(\bar{C}, \bar{d})$ is a Cauchy completion of (M, d) and (b) the d-subspace $(\overline{CM}, \bar{\mathscr{C}}_{\overline{CM}})$ of the d-space $(\bar{C}, \bar{\mathscr{C}})$ is diffeomorphic to $(M, \mathscr{C})$, is said to be a *Cauchy completion of the metric d-space* $(M, \mathscr{C}, d)$.

Proposition 6.3. The d-space $(\overline{CM}, \tilde{\mathscr{C}})$ is the prolonged d-space of $(M, \mathscr{C})$.

Lemma 6.1. Let $(N, \mathscr{D})$ and $(\bar{N}, \bar{\mathscr{D}})$ be d-spaces such that:

(a) N is dense in $\bar{N}$.

(b) $\mathscr{D} = \text{Gen}\, \mathscr{D}_0$, $\bar{\mathscr{D}} = \text{Gen}\, \bar{\mathscr{D}}_0$, where

$$\bar{\mathscr{D}}_0 = \{\bar{\alpha} : \bar{\alpha} : \bar{N} \to \mathbb{R}\} \qquad \text{and} \qquad \mathscr{D}_0 = \{\alpha : \alpha : N \to \mathbb{R}\}$$

(c) $\bar{\mathscr{D}}_0 = \{\bar{\alpha}:$ for every $\alpha \in \mathscr{D}_0$ there is $\bar{\alpha} \in \bar{\mathscr{D}}_0$ such that $\alpha = \bar{\alpha} \,|\, N\}$.

Then $(N, \bar{\mathscr{D}}_N) = (N, \mathscr{D})$.

Proof. The conclusion is a straightforward consequence of the d-subspace definition (see Section 1). ■

Having Definition 6.2, we should learn how to construct the Cauchy completion of a metric d-space $(M, \mathscr{C}, d)$.

Construction 6.1. Let $(M, \mathscr{C}, d)$ be a finitely generated metric d-space, i.e., $\mathscr{C} = \text{Gen}\{\alpha_1, \ldots, \alpha_{n_0}, n_0 \in \mathbb{N}\}$ and $(\bar{C}, \bar{d})$ a Cauchy completion of (M, d). Let us further assume that the following conditions are satisfied, for every $[c] \in \partial M$, $c^k = \{x_n^k : n \in \mathbb{N}\} \in [c]$, $k = 1, 2$, and α_i (for any $i = 1, 2, \ldots, n_0$):

(a) there exist $\lim \alpha_i(x_n^1)$ and $\lim \alpha_i(x_n^2)$

(b) $\lim \alpha_i(x_n^1) = \lim \alpha_i(x_n^2)$.

We define the differential structure $\bar{\mathscr{C}}$ in the following way:

$$\bar{\mathscr{C}} := \operatorname{Gen}\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{n0}\}$$

where $\bar{\alpha}_i: \bar{C} \to \mathbb{R}$ and $\bar{\alpha}_i([c]) := \lim \alpha_i(x_n)$.

Proposition 6.4. The triple $(\bar{C}, \bar{\mathscr{C}}, \bar{d})$ is a Cauchy completion of $(M, \mathscr{C}, d)$ and thus $(\bar{C}, \bar{\mathscr{C}})$ is a prolonged d-space of $(M, \mathscr{C})$.

Proof. The result is a consequence of Lemma 6.1 and Proposition 6.3. ■

As was pointed out in the Introduction, every family of real functions $\mathscr{C}_0$ can be supplemented so as to form a differential structure $\mathscr{C} = \operatorname{Gen} \mathscr{C}_0$. We have used this fact in the above construction. Nothing prevents us from adding new functions to $\mathscr{C}_0$ to obtain another prolongation of the differential structure of the space-time d-space. However, one should remember that not every function, when added to $\mathscr{C}_0$, changes the corresponding differential structure, i.e., it can happen that $\operatorname{Gen} \mathscr{C}_0 = \operatorname{Gen} \mathscr{C}_0'$, where $\mathscr{C}_0' = \mathscr{C}_0 \cup \{f: M \to \mathbb{R}\}$. The condition $\operatorname{Gen} \mathscr{C}_0 \neq \operatorname{Gen} \mathscr{C}_0'$ is satisfied if the function f does not differentially depend on all functions of $\mathscr{C}_0$ at least at one point $p \in M$ (see Appendix B). In the following we shall change the set of generators $\mathscr{C}_0$ of Construction 6.1 by adding to $\mathscr{C}_0$ new functions so as to obtain a set of new generators differentially independent at least at one point of ∂M. In this way we can obtain various nontrivial prolongations of a given differential structure to the singular boundary of space-time.

Construction 6.2. Let

$$\mathscr{C}_0 := \{\alpha_1, \alpha_2, \ldots, \alpha_{n0}\}$$

and

$$\bar{\mathscr{C}}_0 = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{n0}\}$$

be the sets of generators as in Construction 6.1, and let $B = \{\beta : \beta : M \to \mathbb{R}\}$ be a family of real functions satisfying the following conditions:

1. For every $[c] \in \partial M$, $c^1, c^2 \in [c]$, and $\beta \in B$,
   (a) there exist limits $\lim \beta(x_n^1)$, $\lim \beta(x_n^2)$
   (b) $\lim \beta(x_n^1) = \lim \beta(x_n^2)$
2. Every $\beta \in B$ is a smooth function, e.g., $\beta \in \mathscr{C} = \operatorname{Gen} \mathscr{C}_0$.

Moreover, let us define the set $\bar{B} := \{\bar{\beta} : \bar{\beta} : \bar{C} \to \mathbb{R}\}$ such that $\bar{\beta}([c]) := \lim \beta(x_n)$ and for every $\bar{\beta}_0 \in \bar{B}$, $\bar{\beta}_0$ is differentially independent of every function belonging to $\bar{\mathscr{C}}_0 \cup \bar{B} - \{\bar{\beta}_0\}$ at least at one point $[c] \in \partial M$.

We define the following differential structure:

$$\bar{\mathscr{C}}_1 = \mathrm{Gen}(\bar{\mathscr{C}}_0 \cup \bar{B})$$

Proposition 6.5. The triple $(\bar{C}, \bar{\mathscr{C}}_1, \bar{d})$ is a Cauchy completion of $(M, \mathscr{C}, d)$ and consequently $(\bar{C}, \bar{\mathscr{C}})$ is a prolonged d-space of $(M, \mathscr{C})$.

Proof. The proof is analogous to that of Proposition 6.4. ■

Both our constructions start with finitely generated d-spaces since we are interested in prolonging differential structures from space-time manifolds (which are finitely generated d-spaces) to their singular boundaries. However, the general idea of prolongations remains valid for a larger class of metric d-spaces.

7. PROLONGATIONS OF DIFFERENTIAL STRUCTURES TO QUASIREGULAR AND CURVATURE SINGULARITIES

In the present section we continue to study two special, but important, instances, the cosmic string and the closed Friedman world model. Our aim is to construct prolongations of the differential structures of their background d-spaces to their singular boundaries, the first of which consists of quasiregular singularities and the second of curvature singularities.

Example 7.1

Let $\tilde{P}_0 := \mathbb{R}^2 \times \langle 0, \infty) \times \langle 0, 2\pi\rangle$. The background manifold of a cosmic string with singularity defined as a Cauchy boundary is topologically equivalent to $\bar{P} := \tilde{P}_0/\rho_{\mathrm{H}}$ (Gruszczak *et al.*, 1991). Let us introduce a differential structure on $\bar{P}$ with the help of our Construction 6.1.

The set of functions $\bar{\mathscr{P}} := \mathrm{Gen}\{\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_4\}$ is a differential structure, and

$$\bar{\alpha}_i([p_0]) := \begin{cases} \alpha_i([p_0]), & [p_0] \in P \\ \lim\limits_{[p] \to [p_0]} \alpha_i([p]), & [p] \in P, \quad [p_0] \in \partial P \end{cases} \tag{7.1}$$

where $\alpha_i \colon P \to \mathbb{R}$, $i = 0, 1, \ldots, 4$, are defined in Example 2.1. The limit in the above formula exists because $\tilde{\alpha}_i$ does not depend on ϕ at $\rho = 0$.

On the strength of Proposition 6.4, $(\bar{P}, \bar{\mathscr{P}})$ is a prolonged d-space of $(P, \mathscr{P})$. Indeed, $\bar{P}$ is a Cauchy completion of the corresponding metric d-space (Gruszczak *et al.*, 1991), and $\bar{\mathscr{P}}_P = \mathscr{P}$.

Example 7.2

Let us again consider the background manifold of a cosmic string with singularity and Construction 6.2. Let $\tilde{\beta}_1, \ldots, \tilde{\beta}_n, \ldots$ be functions on $\tilde{P}$ defined by

$$\tilde{\beta}_k(t, z, \rho, \phi) := (\{(z^1)^2 + (z^2)^2\}^{1/2})^{m_k/3} = \rho^{m_k/3},$$

where m_k is the k-th prime number greater than 3, and let $\beta_k: P \to \mathbb{R}$, $k = 1, 2, \ldots, n, \ldots$, be the set of real functions defined in the following way:

$$\beta_k([p]) := \tilde{\beta}_k(p), \qquad p \in \tilde{P}_0, \quad [p] \in P$$

The prolonged functions $\bar{\beta}_k: \bar{P} \to \mathbb{R}$ can be obtained from formula (7.1) by making replacements $\alpha \to \beta$ and $i \to k$.

One can check (Proposition B2) that every $\bar{\beta}_k$ is d-independent of the remaining $\bar{\beta}_{k'}$, $k \neq k'$, and $\bar{\alpha}_i$ at $[p] \in \partial P$. Thus, the d-space $(\bar{P}, \bar{\mathscr{P}}_1)$, where

$$\bar{\mathscr{P}}_1 = \mathrm{Gen}(\bar{\alpha}_0, \ldots, \bar{\alpha}_4, \bar{\beta}_1, \ldots, \bar{\beta}_n, \ldots)$$

is infinitely generated and, consequently, can be embedded in no $\mathbb{R}^n$ for any n (Sasin and Żekanowski, 1987).

Example 7.3

Let us consider a prolonged d-space of the background space-time of the closed Friedman world model. We have

$$\tilde{P}_0 := \{p : p = (\eta, \chi, \theta, \phi) \in \langle 0, \pi) \times \langle 0, \pi\rangle \times \langle 0, \pi\rangle \times \langle 0, 2\pi\rangle\}$$

For this model the background manifold with the Cauchy boundary is topologically equivalent to $\bar{P} := \tilde{P}_0/\rho_{\mathrm{H}}$. As above,

$$\bar{\mathscr{P}} := \mathrm{Gen}\{\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_4\}$$

where $\bar{\alpha}_i$ are defined by the formula (7.1), but P, $\mathscr{P}$, and α_i have meanings as in Example 2.2.

$(\bar{P}, \bar{\mathscr{P}})$ is a Cauchy completion of $(P, \mathscr{P})$. Indeed, $\bar{P}$ is a Cauchy completion of the corresponding metric space (Gruszczak, 1990) and $\bar{\mathscr{P}}_P = \mathscr{P}$.

Example 7.4

Let us again consider the closed radiation-filled Friedman world model. Let the real-valued functions $\tilde{\beta}_1, \ldots, \tilde{\beta}_n, \ldots$ and $\beta_1, \ldots, \beta_n, \ldots$ be defined by

$$\tilde{\beta}_k(\eta, \chi, \theta, \phi) := (\{(\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 + (\alpha_4)^2\}^{1/2})^{m_k/3}$$
$$= a_1 \sin^{m_k/3} \eta, \quad m_k \text{ is as above}, \qquad k \in \mathbb{N}$$

$$\beta_k([p]) = \tilde{\beta}_k(p), \qquad p \in \tilde{P}_0, \quad [p] \in P, \quad p = (\eta, \chi. \theta, \phi)$$

As in Example 7.2, we define the prolonged differential structure as $\bar{\mathscr{P}}_{\mathrm{I}} = \mathrm{Gen}(\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_4, \bar{\beta}_1, \ldots, \bar{\beta}_n, \ldots)$. In this way we obtain the d-space $(\bar{P}, \bar{\mathscr{P}}_{\mathrm{I}})$ representing the closed Friedman space-time with singularities which cannot be embedded in any $\mathbb{R}^n$.

Intuitively, we would possibly feel that the Cauchy completion of the Friedman closed space-time of Example 7.3 is "more natural" than that of Example 7.4; however, so far neither mathematical nor physical reasons are known which would allow us to distinguish any of them. Perhaps this fact tells us something about the malicious character of the Friedman singularity.

8. CLASSIFICATION OF DIFFERENTIAL STRUCTURE PROLONGATIONS

The above analyses enable us to give a simple classification of differential structure prolongations to the singular boundary. Although the prolongation procedure is not unique, it would be interesting to know which differential structures can be prolonged to a given singular boundary. Classification of such prolongations would shed light on the nature of the singularities [we adapt to our purposes the classification given by Sasin (1991)].

Let $(M, \mathscr{C})$ be a space-time background d-space (in fact, the manifold) and ∂M its singular boundary. First, we formulate two classification criteria.

Criterion 8.1 (Regularity criterion). A singular boundary point $p \in \partial M$ is called *regular* if there exists a neighborhood $V \in \tau_{\mathscr{C}}$ of p such that the d-space $(V, \mathscr{C}_V)$ has a constant differential dimension n. A boundary point $p \in \partial M$ is called *singular* if p is not regular.[3]

The above definition of regular boundary points is a d-space counterpart of the usual definition of regular singularities. Or, to be more precise, in the traditional classification a singular boundary point is said to be regular if there is an *isometric* extension of the considered space-time M into a larger space-time M' such that the Riemann tensor of M is defined and the singular boundary point is an interior point in M' (Ellis and Schmidt, 1977), whereas in our case only *differential structure* is taken into account (every regular singularity in the traditional sense is regular in our sense). Regular singularities were studied with the help of d-space methods in Heller and Sasin (1991). From the point of view of the d-space theory, regular singularities correspond to d-spaces of class D_0 with constant differential dimension.

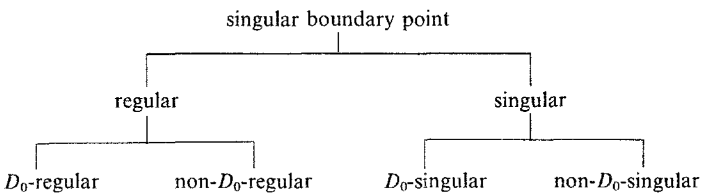
Criterion 8.2 (Local embedding criterion). A boundary point $p \in \partial M$ is said to be of *class* D_0 (D_0-point, for short) if there exists a neighborhood

[3]All boundaries considered in this section are singular boundaries. To avoid cumbersome formulations like "singular singular boundary point," we shall simply speak of boundary points.

$V \in \tau_{\mathscr{C}}$ of p such that $(V, \overline{\mathscr{C}}_V)$ is a d-space of class D_0. A boundary point $p \in \partial M$ is said to be a *non D_0-point* if it is not of class D_0.

Differential spaces of class D_0 constitute an important and easy to handle class of objects. Any Hausdorff d-space which is locally finitely generated is a d-space of class D_0 (Sasin and Żekanowski, 1987), and for a d-space of class D_0 it can be shown (Kowalczyk, 1987) that if ∂M is a set of its singular points, then the set M of its nonsingular points must be open and dense in $\partial M \cup M$.

An immediate outcome of the above definitions is a simple classification scheme of the singular boundary points (or of differential structure prolongations which lead to these boundary points). A regular boundary point can be either D_0 or non-D_0, and similarly a singular boundary point can be either D_0 or non-D_0. In consequence, we have the following diagram:



This diagram deserves a closer analysis. Its upper part looks like the corresponding part of the traditional classification of singular boundary points (Ellis and Schmidt, 1977; Clarke and Schmidt, 1977), but notice that in our case possible prolongations of a given differential structure are taken into account and not the behavior of the curvature as one approaches the singularity.

This is clearly illustrated by our examples. The d-space $(\mathbb{R}, \mathscr{E}_1)$ of Example 4.1 is a D_0-regular prolongation of the d-space $(\mathbb{R}, \mathscr{C}_0)$, whereas the d-space $(\mathbb{R}, \mathscr{C}_2)$ of Example 4.2 is a non-D_0-singular prolongation of $(\mathbb{R}, \mathscr{C}_0)$. The fact that the same d-space has two different prolongations, one of which is regular and another singular, has important consequences. Regular boundary points are removable singularities: one can remove them by isometrically extending the considered space-time in the same way as is done with regular singularities of the traditional classification scheme. However, the singular boundary points are not removable; the differential dimension of such singular points differs from the differential dimension of nonsingular points of space-time, and this is an obstacle to any extension. The above examples show that by suitably prolonging the differential structure of the space-time manifold (recall that this procedure is not unique) we can change the regular boundary point into a singular boundary point (by adding suitable functions to the differential structure—the function $|x|$, in Example 4.2—we change the differential dimension of the singular point).

In connection with the above, three remarks should be made: First, an observer remaining in the interior of space-time is unable to decide whether the singularity he approaches is regular or singular; it depends on the prolongation of the differential structure to the singular boundary, but all such prolongations coincide (by definition) when localized to the space-time manifold. Second, in the case when we have changed a regular boundary point into a singular boundary point by choosing a suitable differential structure prolongation, we are entitled to say that the singularity resides in the differential structure. Third, it cannot be excluded that the future theory of quantum scalar fields on space-time could make use of this "degree of freedom," created by the theory of differential spaces, to form a singularity from an otherwise regular point of space-time. There is no such possibility within the framework of the traditional approach.

Let us consider the d-space of the cosmic string $(P, \mathscr{P})$ of our series of examples. The generator α_4 of its differential structure $\mathscr{P}$ (Example 2.1) is differentially dependent on the generators α_1 and α_2 for any point $p \in P$; however, it becomes differentially independent of them if the generators are prolonged to $\bar{P} - P$ (Example 6.1). This shows that the quasiregular singularity differs from nonsingular points of space-time as far as their differential properties are concerned. Moreover, from Proposition B1 it follows that the local differential dimension of the quasiregular singularity increases by one as compared with nonsingular points of space-time. Therefore, it belongs to the singular boundary and is nonremovable. It is known that in the case of the cosmic string its quasiregular singularity resides in the topology, but it can be strengthened by manipulating the differential structure: the prolongation discussed in Example 6.1 leads to a D_0-singular boundary which is relatively innocuous, but the prolongation discussed in Example 6.2 leads to a non-D_0-singular boundary which cannot be embedded into any finite-dimensional diffferentiable manifold.

Similar remarks concern the curvature singularities of the closed Friedman universe. Such singularities are nonremovable, but the "strength" of the singularity can be modified by suitably choosing the differential structure prolongation: the prolongation of Example 6.3 gives a D_0-singular boundary, and that of Example 6.4 a non-D_0-singular boundary which again cannot be embedded into any finite-dimensional differentiable manifold.

Among our examples there is no d-space with a non-D_0-regular boundary. An example of such a d-space has been constructed by Sasin (1991), but since it looks rather artificial and is not even a space-time, we shall not discuss it here.

9. CONCLUDING REMARKS

In the present work we have reached two main goals: First, we have proposed a global (from the very beginning) description of space-time in terms of differential spaces rather than in terms of differentiable manifolds. Second, we have discussed prolongations of the differential structure of a given space-time to its singular boundary. The first achievement is an improvement of known global methods; the second one sheds new light on the complicated nature of singularities. As far as we know, this problem had never been investigated.

Many attempts have focused on topological relations between space-time and its singular boundary; we have demonstrated that, even in the cases of regular and quasiregular singularities, the differential space modeling a space-time with such singularities can have the nontrivial property of being nonembeddable in any $\mathbb{R}^n$.

It turns out that if there is no noncompatibility of topologies, the differential structure can be prolonged to singularities. However, this procedure is not unique. For the space-time of a cosmic string and of closed Friedman universes, we have constructed prolongations of their differential structures to the singular boundaries. Some of these prolongations lead to differential spaces which can be embedded into $(\mathbb{R}^n, \mathscr{E}_n)$ for some $n<\infty$, and some to differential spaces which can be embedded in no such Euclidean differential space.

This nonuniqueness of differential structure prolongations contributes to a better understanding of the nature of singularities. Singularities can admit many nontrivial differential structures. One cannot exclude that a future theory, e.g., a theory of quantum gravity or a theory of quantum fields on a curved background space-time, would in some way distinguish one such admissible differential structure (it should be remembered that functions belonging to a differential structure can be interpreted as scalar fields on a differential space). The classification of admissible prolongations of differential structures to singularities proposed in the present work could be regarded as a first small step in this direction.

APPENDIX A. TOPOLOGY ON d-SPACES

Let $\mathscr{C}$ be a family of continuous real functions on a topological space M. The space M is said to be *$\mathscr{C}$-regular* if for every open set $U\subset M$ and for every point $p\in U$ there exists a function $f\in\mathscr{C}$ separating p in U, i.e., a function which equals 1 on a neighborhood V of p and equals 0 on an open set V_0 which together with the set U forms the covering of M.

We shall show that any topological space $(M, \tau_{\mathscr{C}})$ is $\mathscr{C}$-regular. Indeed, let U be an open set in M, and $p\in U$. There is a mapping $f=(f^1, \ldots, f^n)\colon M\to\mathbb{R}^n$, where $f^1, \ldots, f^n\in\mathscr{C}$, and an open set $P\subseteq\mathbb{R}^n$ such that $p\in f^{-1}(P)\subseteq U$. Here $\mathbb{R}^n$ is $\mathscr{E}$-regular, where $\mathscr{E}$ is the set of all smooth functions on $\mathbb{R}^n$ [for details see Sikorski (1972)].

Let us recall that a topological space $(M, \tau_{\mathscr{C}})$ is said to be T_0 if for every pair of its disjoint points there is an open set containing only one of these points. A topological space $(M, \tau_{\mathscr{C}})$ is said to be $T_{3\frac{1}{2}}$ (or *Tichonov*) if it is T_1 and if for any point $p \in M$ and a closed set $F \subset M$ which does not contain p there exists a continuous function f such that $f(p)=0$ and $f(q)=1$ for $q \in F$. We say $(M, \mathscr{C})$ is T_1 if for every pair of its different points $x, y \in M$ there exists an open set U such that $x \in U$ and $y \neq U$ (Hausdorff topological space is called T_2).

Now we shall show that, for any d-space $(M, \mathscr{C})$, if $(M, \tau_{\mathscr{C}})$ is T_0, then it is $T_{3\frac{1}{2}}$. Indeed, for any $x, y \in M$, $x \neq y$, there is a neighborhood U_1 of x such that $y \notin U_1$. Since $(M, \mathscr{C})$ is $\mathscr{C}$-regular, there is $f \in \mathscr{C}$ such that $f^{-1}(\frac{2}{3}, 2)$ and $f^{-1}(-1, \frac{1}{3})$ are disjoint neighborhoods of x and y, respectively.

APPENDIX B. DIFFERENTIAL DIMENSION

The theory of differential dimensions has been developed in Multarzyński and Sasin (1989) and Heller *et al.* (1991); here we only outline the basic concepts. In the following, the *local differential dimension* of a differential space $(M, \mathscr{C})$ is understood as a dimension of the tangent space $T_p M$ to $(M, \mathscr{C})$ at $p \in M$. However, one should notice that in the case of a d-space its differential dimension can change from point to point, and it can be different from its topological dimension. To make this precise, we introduce the following concepts.

Definition B1. A function $f \in \mathscr{C}$ is said to be *differentially dependent* (briefly, *d-dependent*) on functions $g_1, \ldots, g_n \in \mathscr{C}$ at a point $p \in M$ if there exist a neighborhood $U \in \tau_{\mathscr{C}}$ of the point p and a function $\omega \in \mathscr{E}_n$ such that

$$f\,|_U = \omega \circ (g_1, \ldots, g_n)\,|_U$$

Definition B2. A set $\{f_1, \ldots, f_n\} \subset \mathscr{C}$ is said to be *differentially independent* (*d-independent*) at a point $p \in M$ if no function f_i, for $i \in \{1, 2, \ldots, n\}$, differentially depends on the remaining functions at p. Any set $\mathscr{F} \subset \mathscr{C}$ is said to be *differentially independent* at $p \in M$ if every subset of $\mathscr{F}$ is d-independent at p. A set $\mathscr{F} \subset \mathscr{C}$ is said to be *d-dependent* at $p \in M$ if $\mathscr{F}$ is not d-independent at p.

Both d-dependence and d-independence of a set $\mathscr{F}$ are local properties of this set.

It is easy to see that the set $\{\pi_1, \ldots, \pi_n\} \subset \mathscr{E}_n$ of projections into coordinate axes is d-independent at any point $p \in \mathbb{R}^n$. It can also be shown that if $M \subset \mathbb{R}^n$ is a k-dimensional hypersurface, then the set of projections $\{\pi_1|_M, \ldots, \pi_n|_M\} \subset C^\infty(M)$ is d-independent at an arbitrary point $p \in M$ if and only if $k=n$. From this it follows that if $M \subset \mathbb{R}^n$ is a non-empty subset and $\mathscr{C}=\mathscr{E}_{nM}$, then the set of the projections $\{\pi_1|_M, \ldots, \pi_n|_M\}$ is d-independent at $p\in M$ if and only if dim $T_pM=n$.

Proposition B1. Let $(M, \mathscr{C})$ be a d-space with $\mathscr{C}=\text{Gen}\{f_1, \ldots, f_n\}$. The set of functions $\{f_1, \ldots, f_n\}\subset\mathscr{C}$ is d-independent at $p\in M$ if and only if dim $T_pM=n$.

Another useful characterization of the d-independence of a set of real functions belonging to $\mathscr{C}$ is given by the following.

Proposition B2. Let $(M, \mathscr{C})$ be a d-space. A subset $\{f_1, \ldots, f_n\}$ of $\mathscr{C}$ is d-independent at a point $p\in M$ if and only if, for any function $\omega\in\mathscr{E}_n$ and any neighborhood $U\in\tau_{\mathscr{C}}$ of the point p, the following implication is true:

$$\omega\circ(f_1, \ldots, f_n)|_U=0 \Leftrightarrow \forall 1\leq i\leq n, \qquad \omega'|_i(f_1(p), \ldots, f_n(p))=0$$

From Proposition B1 one has the following result:

Corollary B1. If tangent vectors $v_1, \ldots, v_n\in T_pM$ are linearly independent, then any set of functions $\{f_1, \ldots, f_n\}\subset\mathscr{C}$ such that

$$v_i(f_j)=\delta_{ij} \qquad \text{for} \quad i, j=1, 2, \ldots, n \tag{$*$}$$

is d-independent at p.

REFERENCES

Aronszajn, N. (1967). *Notices of the American Mathematical Society*, **14**, 111.
Bosshard, B. (1976). *Communications in Mathematical Physics*, **46**, 263.
Clarke, C. J. S. (1970). *Proceedings of the Royal Society London A*, **314**, 417.
Clarke, C. J. S. (1978). *Communications in Mathematical Physics*, **58**, 291.
Clarke, C. J. S., and Schmidt, B. G. (1977). *General Relativity and Gravitation*, **8**, 129.
Dodson, C. T. J. (1978). *International Journal of Theoretical Physics*, **17**, 389.
Dodson, C. T. J. (1979). *General Relativity and Gravitation*, **10**, 969.
Ellis, G. F. R., and Schmidt, B. G. (1977). *General Relativity and Gravitation*, **8**, 915.
Geroch, R. (1972). *Communications in Mathematical Physics*, **26**, 271.
Gott III, J. R. (1985). *Astrophysical Journal*, **288**, 422.
Gruszczak, J. (1990). *International Journal of Theoretical Physics*, **29**, 37.
Gruszczak, J., Heller, M., and Multarzyński, P. (1988). *Journal of Mathematical Physics*, **29**, 2576.
Gruszczak, J., Heller, M., and Multarzyński, P. (1989). *Foundations of Physics*, **19**, 607.
Gruszczak, J., Heller, M., and Pogoda, Z. (1991). *International Journal of Theoretical Physics*, **30**, 555.
Gruszczak, J., Heller, M., and Sasin, W. (1992). *Acta Cosmologica*, **18**, 45–55.
Hawking, S. W., and Ellis, G. F. R. (1973). *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge.
Heller, M. (1992). *International Journal of Theoretical Physics*, **31**, 277.
Heller, M., and Sasin, W. (1991). *Acta Cosmologica*, **17**, 7.
Heller, M., Multarzyński, P., and Sasin, W. (1989). *Acta Cosmologica*, **16**, 53.
Heller, M., Multarzyński, P., Sasin, W., and Żekanowski, Z. (1991). *Acta Cosmologica*, **17**, 19.

Heller, M., Sasin, W., Trafny, A., and Zekanowski, Z. (1992). *Acta Cosmologica*, **18**, 57.
Hiscock, W. A. (1985). *Physical Review D*, **31**, 3288.
Isham, C. J. (1989). *Modern Differential Geometry for Physicists*, World Scientific, Singapore.
Johnson, R. A. (1977). *Journal of Mathematical Physics*, **18**, 898.
Kibble, T. (1976). *Journal of Physics A*, **9**, 1387.
Kowalczyk, A. (1987). Differential spaces and manifolds, preprint, Telecom Australia Research Laboratories, Clayton.
Levi-Civita, T. (1917). *Accademia Nazionale dei Lincei Rendiconti*, **26**, 308.
MacLane, S. (1970). Differential spaces. Notes for geometrical mechanics, unpublished.
Marshall, C. D. (1975). *Journal of Differential Geometry*, **10**, 551.
Mostov, M. A. (1979). *Journal of Differential Geometry*, **14**, 255.
Multarzyński, P., and Sasin, W. (1989). *Supplemento ai Rendiconti del Circolo Matematico di Palermo, Ser. II*, **22**, 193.
Sasin, W. (1988). *Commentationes Mathematicae Universitatis Carolinae*, **29**, 529.
Sasin, W. (1991). *Demonstratio Mathematica*, **24**, 601.
Sasin, W., and Żekanowski, Z. (1987). *Demonstratio Mathematica*, **20**, 477.
Schmidt, B. G. (1971). *General Relativity and Gravitation*, **1**, 269.
Sikorski, R. (1967). *Colloquium Mathematicum*, **18**, 251.
Sikorski, R. (1971). *Colloquium Mathematicum*, **24**, 45.
Sikorski, R. (1972). *Introduction to Differential Geometry*, Polish Scientific Publishers, Warsaw [in Polish].
Spallek, K. (1969). *Mathematische Annalen*, **180**, 269.
Staruszkiewicz, A. (1963). *Acta Physica Polonica*, **24**, 734.
Tipler, F. J., Clarke, C. J. S., and Ellis, G. F. R. (1980). In *General Relativity and Gravitation*, Vol. 2, A. Held, ed., Plenum Press, New York, pp. 97–206.
Vickers, J. A. G. (1985). *Classical and Quantum Gravity*, **2**, 755.
Vickers, J. A. G. (1987). *Classical and Quantum Gravity*, **4**, 1.
Vickers, J. A. G. (1990). *Classical and Quantum Gravity*, **7**, 731.
Vilenkin, A. (1981*a*). *Physical Review D*, **24**, 2082.
Vilenkin, A. (1981*b*). *Physical Review Letters*, **46**, 1169.